

Equivariant Cyclic Cohomology of H-Algebras

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Abstract

We define an equivariant K_0 -theory for *Yetter-Drinfeld* algebras over a Hopf algebra with an invertible antipode. We then show that this definition can be generalized to all Hopf-module algebras. We show that there exists a pairing, generalizing Connes' pairing, between this theory and a suitably defined Hopf algebra equivariant cyclic cohomology theory.

Keywords. Cyclic cohomology, Hopf algebras, equivariant K-theory.

1 Introduction

Equivariant cyclic cohomology, for actions of discrete groups or compact Lie groups on algebras, has been studied by various authors [2, 3, 8, 7, 15, 16]. One of the main themes studied in these papers is the relation between equivariant cyclic cohomology and the cyclic cohomology of the corresponding crossed product algebra. In [1], we extended one of the main results of these investigations, namely the Feigin-Tsygan and (independently) Nistor spectral sequence [7, 15] to actions of Hopf algebras. The E_2 -term of this spectral sequence can be considered as the complex of noncommutative equivariant de Rham cochains on the given algebra.

One of the main features of cyclic cohomology is the existence of a pairing, through Connes' Chern character, between K -theory and cyclic cohomology. In attempting to extend this pairing to a Hopf algebra equivariant setting one faces the following problem. Let \mathcal{H} be a Hopf algebra. For an \mathcal{H} -module algebra A , here called an \mathcal{H} -algebra, and a finite dimensional \mathcal{H} -module V , one would like a natural algebra structure on $A \otimes \text{End}(V)$ together with a natural \mathcal{H} -action, to turn it into an \mathcal{H} -algebra. There seems to be two possible approaches to this problem, depending on whether we use the diagonal \mathcal{H} -action or not. In Section 6 we show that the two definitions in fact coincide. If we use the diagonal \mathcal{H} -action on $A \otimes \text{End}(V)$, then the tensor product algebra structure on $A \otimes \text{End}(V)$ is not an \mathcal{H} -algebra, unless \mathcal{H} is cocommutative. In general, one has to twist this tensor product structure with the help of an extra structure on A . This problem is naturally solved by introducing the class of Yetter-Drinfeld algebras over a Hopf algebra.

One can then show that for a Yetter-Drinfeld algebra A , $A \otimes \text{End}(V)$ has a natural \mathcal{H} -algebra structure. In fact we first discovered formula (9) in Lemma 5.2 and realized later on that this condition is equivalent to a well known condition, namely the Yetter-Drinfeld condition, provided the antipode of \mathcal{H} is bijective. We define the equivariant K_0 -theory, $K_0^{\mathcal{H}}(A)$, of a Yetter-Drinfeld algebra A as the Grothendieck group of the semigroup of invariant idempotents in $A \otimes \text{End}(V)$ for all finite dimensional \mathcal{H} -modules V .

Alternatively, one can define a non-diagonal \mathcal{H} -action on $A \otimes \text{End}(V)$ by embedding it into $\text{End}(A \otimes V)$ first and then induce the conjugation \mathcal{H} -action as in [14] (see formula (12)). One can then check that endowed with the tensor product algebra structure, $A \otimes \text{End}(V)$ is an \mathcal{H} -algebra. This approach leads to an apparently different definition of equivariant K -theory [14] that also admits a pairing with the equivariant cyclic cohomology defined in this paper. In Section 6, motivated by examples from (co)quasitriangular Hopf algebras, we show that the two definitions in fact coincide. We should also mention that the trace map used in [14] is exactly the trace map introduced in an earlier version of the present paper for cocommutative Hopf algebras. An interesting feature in our generalization of Connes' Chern character which is patterned after Connes' original construction in [6] is the equivariant trace map Ψ (Proposition 5.2). We remark that the complex of cyclic equivariant cochains introduced in Section 3 is not quite the same as the complex that naturally appeared in [1]. We can prove, however, that it enjoys the same relation to crossed product algebras (Theorem 4.2). This complex behaves better with respect to pairing with K -theory and this motivated our choice.

2 Preliminaries

In this paper we work over a fixed field k of characteristic zero. We denote the coproduct, antipode and counit of a Hopf algebra by Δ , S and ϵ , respectively. Let \mathcal{H} be a Hopf algebra. We use Sweedler's notation and write $\Delta h = h^{(0)} \otimes h^{(1)}$, where summation is understood. Similarly, we write $\Delta^{(n)} h = h^{(0)} \otimes h^{(1)} \otimes \cdots \otimes h^{(n)}$, where $\Delta^{(n)} : \mathcal{H} \rightarrow \mathcal{H}^{\otimes(n+1)}$ is defined by $\Delta^{(1)} = \Delta$ and $\Delta^{(n)} = (\Delta \otimes 1) \circ \Delta^{(n-1)}$, $n \geq 2$. By a left \mathcal{H} -module we mean a left \mathcal{H} -module over the underlying algebra of \mathcal{H} . Let A be an algebra. We say A is a left \mathcal{H} -algebra, also called a left \mathcal{H} -module algebra, if A is a left \mathcal{H} -module and for all $a, b \in A, h \in \mathcal{H}$,

$$\begin{aligned} h \cdot (ab) &= (h^{(0)} \cdot a)(h^{(1)} \cdot b), \\ h \cdot 1 &= \epsilon(h)1. \end{aligned}$$

By a *paracocyclic* object in a category \mathcal{A} [7, 8] we mean a cosimplicial object A in \mathcal{A} endowed with operators $\tau_n : A_n \rightarrow A_n$, called cyclic operators, such that the following extra relations are satisfied:

$$\begin{aligned} \tau_{n+1} \partial^i &= \partial^{i-1} \tau_n, & 1 \leq i \leq n, & \quad \tau_{n+1} \partial^0 = \partial^{n+1}, \\ \tau_{n-1} \sigma^i &= \sigma^{i-1} \tau_n, & 1 \leq i \leq n, & \quad \tau_{n-1} \sigma^0 = \sigma^{n-1} \tau_n^2, \end{aligned} \tag{1}$$

where $\partial^i : A_n \rightarrow A_{n+1}$ are coface maps and $\sigma^i : A_n \rightarrow A_{n-1}$ are codegeneracies. We note that a similar notion is independently introduced in [15, 16]. If in addition we have $\tau_n^{n+1} = id$ for all $n \geq 0$, then we have a *cocyclic* object in the sense of Connes [5]. By a *bi-paracocyclic* object in \mathcal{A} , we mean a paracocyclic object in the category of paracocyclic objects in \mathcal{A} . So, giving a bi-paracocyclic object in \mathcal{A} is equivalent to giving a double sequence $A(p, q)$ of objects of \mathcal{A} and operators $\partial_{p,q}, \sigma_{p,q}, \tau_{p,q}$ and $\bar{\partial}_{p,q}, \bar{\sigma}_{p,q}, \bar{\tau}_{p,q}$ such that, for all $p \geq 0$,

$$B_p(q) = \{A(p, q), \sigma_{p,q}^i, \partial_{p,q}^i, \tau_{p,q}\},$$

and for all $q \geq 0$,

$$\bar{B}_q(p) = \{A(p, q), \bar{\sigma}_{p,q}^i, \bar{\partial}_{p,q}^i, \bar{\tau}_{p,q}\},$$

are paracocyclic objects in \mathcal{A} and every horizontal operator commutes with every vertical operator.

We say that a bi-paracocyclic object is *cocylindrical* [8] if for all $p, q \geq 0$,

$$\bar{\tau}_{p,q}^{p+1} \tau_{p,q}^{q+1} = id_{p,q}. \quad (2)$$

If A is a bi-paracocyclic object in \mathcal{A} , the paracocyclic object related to the diagonal of A will be denoted by ΔA . So, the paracocyclic operators on $\Delta A(n) = A(n, n)$ are $\bar{\partial}_{n,n+1}^i \partial_{n,n}^i, \bar{\sigma}_{n,n-1}^i \sigma_{n,n}^i, \bar{\tau}_{n,n} \tau_{n,n}$. When A is cocylindrical, since the cyclic operator of ΔA is $\bar{\tau}_{n,n} \tau_{n,n}$ and $\bar{\tau}, \tau$ commute, then, from $\bar{\tau}_{n,n}^{n+1} \tau_{n,n}^{n+1} = id_{n,n}$, we conclude that $(\bar{\tau}_{n,n} \tau_{n,n})^{n+1} = id$. So that ΔA is a cocyclic object.

A *paracochain complex* [8], by definition, is a graded k -module $\mathbf{V}^\bullet = (V^i)_{i \geq 0}$ equipped with operators $b : V^i \rightarrow V^{i+1}$ and $B : V^i \rightarrow V^{i-1}$ such that $b^2 = B^2 = 0$, and the operator $T = 1 - (bB + Bb)$ is invertible. In the case that $T = 1$, the paracochain complex is called a *mixed complex*.

Corresponding to any paracocyclic module A , we can define the paracochain complex $\mathbf{C}^\bullet(A)$ with the underlying graded module $\mathbf{C}^n(A) = A(n)$ and the operators $b = \sum_{i=0}^n (-1)^i \partial^i$ and $B = N\sigma(1 - (-1)^{n+1}\tau)$. Here, σ is the extra degeneracy satisfying $\tau\sigma^0 = \sigma\tau$, and $N = \sum_{i=0}^n (-1)^{in} \tau^i$ is the norm operator. For any bi-paracocyclic module A , $Tot(\mathbf{C}(A))$ is a paracochain complex with $Tot^n(\mathbf{C}(A)) = \sum_{p+q=n} A(p, q)$ and with the operators $Tot(b) = b + \bar{b}$ and $Tot(B) = B + T\bar{B}$, where $T = 1 - (bB + Bb)$. It is a mixed complex if A is cocylindrical [8].

If we define the normalized cochain functor \mathbf{N} from paracocyclic modules to paracochain complexes with the underlying graded module $\mathbf{N}^n(A) = \bigcap_{i=0}^{n-1} \ker(\sigma^i)$ and the operators b, B induced from $\mathbf{C}^\bullet(A)$, then we have the following well-known results (see [8] for a dual version):

1. The inclusion $(\mathbf{N}^\bullet(A), b) \rightarrow (\mathbf{C}^\bullet(A), b)$ is a quasi-isomorphism of complexes.
2. The cyclic Eilenberg-Zilber theorem holds for cocylindrical modules, i.e., for any cocylindrical module A , there is a natural quasi-isomorphism $\mathbf{f}_0 + \mathbf{uf}_1 : \mathbf{N}^\bullet(\Delta(A)) \rightarrow Tot^\bullet(\mathbf{N}(A))$ of mixed complexes, where \mathbf{f}_0 is the shuffle map.

3 Equivariant cyclic cohomology of \mathcal{H} -algebras

In this section we introduce the complex of cyclic equivariant cochains for \mathcal{H} -algebras. It is a noncommutative analogue of the complex of equivariant differential forms (Cartan model, see e.g. [2]). Since compact quantum groups naturally coact on interesting algebras like quantum spheres, it would be perhaps more natural to consider Hopf comodule algebras. Passing to this dual setting does not present serious difficulties. Our cocyclic module in Theorem 3.1 is not quite the dual of the cyclic module that appeared in the E^2 -term of the spectral sequence in [1], but is very similar to it. In particular, Theorem 4.2 in the next section shows that this version of equivariant cyclic cohomology enjoys the same relation with cyclic cohomology of crossed product algebras as in the main theorem of [1]. The reason we prefer the present complex is that it works better for pairing with K -theory.

Let \mathcal{H} be a Hopf algebra with a bijective antipode and let $F(\mathcal{H})$ be the space of k -linear maps $f : \mathcal{H} \rightarrow k$. Let A be an \mathcal{H} -algebra and let $C^n(A, F(\mathcal{H}))$ denote the linear space of $(n+1)$ -linear mappings

$$f : A^{\otimes(n+1)} = A \otimes A \otimes \cdots \otimes A \rightarrow F(\mathcal{H}).$$

We define an \mathcal{H} -action on $C^n(A, F(\mathcal{H}))$ by

$$(h \cdot f)(a_0, a_1, \dots, a_n)(g) = f(h^{(0)} \cdot a_0, h^{(1)} \cdot a_1, \dots, h^{(n)} \cdot a_n)(g) \quad h, g \in \mathcal{H}, a_i \in A \quad (3)$$

A cochain $f \in C^n(A, F(\mathcal{H}))$ is called \mathcal{H} -equivariant if for all $h, g \in \mathcal{H}, a_i \in A$,

$$\begin{aligned} h \cdot f(a_0, \dots, a_n)(g) &= f(a_0, \dots, a_n)(S^{-1}(h) \cdot g) \\ &= f(a_0, \dots, a_n)(S(h^{(1)})gh^{(0)}), \end{aligned} \quad (4)$$

where the action on the left is defined in (3) and the action on the right is defined by

$$h \cdot g = S^2(h^{(0)})gS(h^{(1)}),$$

so that $S^{-1}(h) \cdot g = S(h^{(1)})gh^{(0)}$. We define $C_{\mathcal{H}}^n(A)$ to be the linear space of all \mathcal{H} -equivariant $f \in C^n(A, F(\mathcal{H}))$.

We define a cocyclic module structure on the spaces $\{C_{\mathcal{H}}^n(A)\}_{n \geq 0}$. First we define the cyclic operator T_n on $C_{\mathcal{H}}^n(A)$ by

$$T_n f(a_0, a_1, \dots, a_n)(g) := f(S^{-1}(g^{(0)}) \cdot a_n, a_0, \dots, a_{n-1})(g^{(1)}). \quad (5)$$

Lemma 3.1. T_n is an equivariant map, i.e., $T_n f \in C_{\mathcal{H}}^n(A)$, for $f \in C_{\mathcal{H}}^n(A)$.

Proof. We should check that $h \cdot T_n f(g) = T_n f(S^{-1}(h) \cdot g)$:

$$\begin{aligned}
& T_n f(a_0, a_1, \dots, a_n)(S^{-1}(h) \cdot g) \\
&= T_n f(a_0, a_1, \dots, a_n)(S(h^{(1)})gh^{(0)}) \\
&= f((S^{-1}(h^{(0)})S^{-1}(g^{(0)})h^{(3)}) \cdot a_n, a_0, \dots, a_{n-1})(S(h^{(2)})g^{(1)}h^{(1)}) \\
&= f((S^{-1}(h^{(0)})S^{-1}(g^{(0)})h^{(2)}) \cdot a_n, a_0, \dots, a_{n-1})(S^{-1}(h^{(1)}) \cdot g^{(1)}) \\
&= h^{(1)} \cdot f((S^{-1}(h^{(0)})S^{-1}(g^{(0)})h^{(2)}) \cdot a_n, a_0, \dots, a_{n-1})(g^{(1)}) \\
&= f((h^{(1)}S^{-1}(h^{(0)})S^{-1}(g^{(0)})h^{(n+2)}) \cdot a_n, h_{(2)} \cdot a_0, \dots, h^{(n+1)} \cdot a_{(n-1)})(g^{(1)}) \\
&= f(S^{-1}(g^{(0)}) \cdot (h^{(n)} \cdot a_n), h^{(0)} \cdot a_0, \dots, h^{(n-1)} \cdot a_{n-1})(g^{(1)}) \\
&= T_n f(h^{(0)} \cdot a_0, \dots, h^{(n)} \cdot a_n)(g) \\
&= h \cdot T_n f(a_0, \dots, a_n)(g).
\end{aligned}$$

□

We define the coface and codegeneracy operators on $C_{\mathcal{H}}^n(A)$ as follows:

$$\partial^i : C_{\mathcal{H}}^{n-1}(A) \rightarrow C_{\mathcal{H}}^n(A), \quad \sigma^i : C_{\mathcal{H}}^{n+1}(A) \rightarrow C_{\mathcal{H}}^n(A),$$

$$\begin{aligned}
\partial^i f(a_0, \dots, a_n)(g) &= f(a_0, \dots, a_i a_{i+1}, \dots, a_n)(g), \quad 0 \leq i \leq n-1, \\
\partial^n f(a_0, \dots, a_n)(g) &= f((S^{-1}(g^{(0)}) \cdot a_n) a_0, a_1, \dots, a_{n-1})(g^{(1)}), \\
\sigma^i f(a_0, \dots, a_{n-1}, a_n)(g) &= f(a_0, \dots, a_i, 1, a_{i+1}, \dots, a_n)(g), \quad 0 \leq i \leq n.
\end{aligned} \tag{6}$$

One can check that these operators are well defined, i.e., they send equivariant cochains to equivariant cochains. Now we are ready to state the main result of this section.

Theorem 3.1. *For any Hopf algebra \mathcal{H} with a bijective antipode and an \mathcal{H} -algebra A , the \mathcal{H} -equivariant space $C_{\mathcal{H}}^{\natural}(A) = \{C_{\mathcal{H}}^n(A)\}_{n \geq 0}$ with operators defined in (6), is a cocyclic module.*

Proof. We only check those identities that involve the cyclic operator T and leave the rest to the reader.

- $T_n \partial^0 = \partial^n.$

$$\begin{aligned}
& (T_n \partial^0 f)(a_0, a_1, \dots, a_n)(g) = \partial^0 f(S^{-1}(g^{(0)}) \cdot a_n, a_0, \dots, a_{n-1})(g^{(1)}) \\
&= f((S^{-1}(g^{(0)}) \cdot a_n) a_0, \dots, a_{n-1})(g^{(1)}) = \partial^n f(a_0, a_1, \dots, a_n)(g).
\end{aligned}$$

- $T_n \partial^i = \partial^{i-1} T_{n+1}.$ For $1 \leq i < n$, this is obvious. For $i = n$, we have

$$\begin{aligned}
& (T_n \partial^n f)(a_0, a_1, \dots, a_n)(g) = \partial^n f(S^{-1}(g^{(0)}) \cdot a_n, a_0, \dots, a_{n-1})(g^{(1)}) \\
&= f((S^{-1}(g^{(1)}) \cdot a_{n-1})(S^{-1}(g^{(0)}) \cdot a_n), a_0, \dots, a_{n-2})(g^{(2)}) \\
&= f(S^{-1}(g^{(0)}) \cdot (a_{n-1} a_n), a_0, \dots, a_{n-2})(g^{(1)}) \\
&= T_n f(a^0, \dots, a^{n-2}, a^{n-1} a^n)(g) = (\partial^{n-1} T_{n+1} f)(a_0, \dots, a_n)(g).
\end{aligned}$$

- $T_n \sigma^0 = \sigma^n T_{n+1}^2.$

$$\begin{aligned}
& (\sigma^n T_{n+1}^2 f)(a_0, \dots, a_n)(g) \\
&= (T_{n+1}^2 f)(a_0, \dots, a_n, 1)(g) = T_{n+1} f((S^{-1}(g^{(0)}) \cdot 1), a_0, \dots, a_n)(g^{(1)}), \\
&= f((S^{-1}(g^{(1)}) \cdot a_n), (S^{-1}(g^{(0)}) \cdot 1), a_0, \dots, a_{n-1})(g^{(2)}), \\
&= f(S^{-1}(g^{(0)}) \cdot a_n, 1, a_0, \dots, a_{n-1})(g^{(1)}) = (\sigma^0 f)(S^{-1}(g^{(0)}) \cdot a_n, a_0, \dots, a_{n-1})(g^{(1)}) \\
&= (T_n \sigma^0 f)(a_0, a_1, \dots, a_{n-1}, a_n)(g).
\end{aligned}$$

- $T_n \sigma^i = \sigma^{i-1} T_{n+1}, \quad 1 \leq i \leq n.$ This is obvious.

- $T_n^{n+1} = \text{id}_n.$ Let $f \in C_{\mathcal{H}}^n(A).$ Then,

$$\begin{aligned}
T_n f(a_0, a_1, \dots, a_n)(g) &= f(S^{-1}(g^{(0)}) \cdot a_n, \dots, a_{n-1})(g^{(1)}) \\
T_n^2 f(a_0, a_1, \dots, a_n)(g) &= f(S^{-1}(g^{(1)}) \cdot a_{n-1}, S^{-1}(g^{(0)}) \cdot a_n, \dots, a_{n-2})(g^{(2)}) \\
&\vdots \\
T_n^n f(a_0, a_1, \dots, a_n)(g) &= f(S^{-1}(g^{(n-1)}) \cdot a_1, S^{-1}(g^{(n-2)}) \cdot a_2, \dots, S^{-1}(g^{(0)}) \cdot a_n, a_0)(g^{(n)}).
\end{aligned}$$

Thus,

$$\begin{aligned}
& T_n^{n+1} f(a_0, a_1, \dots, a_n)(g) \\
&= f(S^{-1}(g^{(n)}) \cdot a_0, S^{-1}(g^{(n-1)}) \cdot a_1, \dots, S^{-1}(g^{(0)}) \cdot a_n)(g^{(n+1)}) \\
&= S^{-1}(g^{(0)}) \cdot f(a_0, a_1, \dots, a_n)(g^{(1)}) = f(a_0, a_1, \dots, a_n)(S^{-2}(g^{(0)}) \cdot g^{(1)}) \\
&= f(a_0, a_1, \dots, a_n)(g^{(0)} g^{(2)} S^{-1}(g^{(1)})) = f(a_0, a_1, \dots, a_n)(g^{(0)} \epsilon(g^{(1)})) \\
&= f(a_0, a_1, \dots, a_n)(g).
\end{aligned}$$

This last identity completes our proof of the theorem. \square

We denote the Hochschild, cyclic, and periodic cyclic cohomology groups of the cocyclic module $\{C_{\mathcal{H}}^n(A)\}_{n \geq 0}$ by $HH_{\mathcal{H}}^{\bullet}(A)$, $HC_{\mathcal{H}}^{\bullet}(A)$, and $HP_{\mathcal{H}}^{\bullet}(A)$, respectively.

Example 1.(trivial actions) Assume that \mathcal{H} acts trivially on A , i.e., $h \cdot a = \epsilon(h)a$ for all $h \in \mathcal{H}, a \in A$. Then the cocyclic module $\{C_{\mathcal{H}}^n(A)\}_{n \geq 0}$ simplifies as follows. We have $C_{\mathcal{H}}^n(A) \simeq C^n(A) \otimes R(\mathcal{H})$, where $R(\mathcal{H}) \subset F(\mathcal{H})$ is the space of invariant linear functionals on \mathcal{H} . By definition, $f \in R(\mathcal{H})$ if $f(S^{-1}(h) \cdot g) = \epsilon(h)f(g)$ for all $h, g \in \mathcal{H}$. It then follows that $HC_{\mathcal{H}}^n(A) \simeq HC^n(A) \otimes R(\mathcal{H}), n \geq 0$.

Example 2.(Morita invariance) Let A be a left \mathcal{H} -algebra. Then the algebra of $r \times r$ matrices over A , $M_r(A) = A \otimes M_r(k)$, is a left \mathcal{H} -algebra where the left \mathcal{H} -action is defined by $h \cdot (a \otimes m) = h \cdot a \otimes m$. The *equivariant trace map* $tr : C_{\mathcal{H}}^n(A) \rightarrow C_{\mathcal{H}}^n(M_r(A)), n \geq 0$, is defined by

$$(tr f)(a_0 \otimes m_0, \dots, a_n \otimes m_n)(g) = tr(m_0 \dots m_n) f(a_0, \dots, a_n)(g),$$

where tr is the usual trace on $M_r(k)$. It can be checked that tr is a morphism of cocyclic modules. It follows from Corollary 5.1 that the induced map $tr : HC_{\mathcal{H}}^n(A) \rightarrow HC_{\mathcal{H}}^n(M_r(A))$ is an isomorphism for $n \geq 0$.

Example 3. Let $\theta : A \rightarrow A$ be an automorphism of an algebra A . Then A is an $k[x, x^{-1}]$ -module algebra, where $k[x, x^{-1}]$ is the Hopf algebra of Laurent polynomials. We identify the equivariant cyclic complex as follows. We have an isomorphism

$$Hom(k[x, x^{-1}] \otimes A^{\otimes(n+1)}, k) \simeq \prod_{-\infty}^{+\infty} Hom(A^{\otimes(n+1)}, k),$$

sending $f \mapsto (f_m)_{-\infty}^{+\infty}$, where $f_m(a_0, \dots, a_n) = f(x^m, a_0, \dots, a_n)$. It is clear that a cochain $(f_m)_{-\infty}^{+\infty}$ is equivariant iff for all m

$$f_m(\theta a_0, \theta a_1, \dots, \theta a_n) = f_m(a_0, \dots, a_n) \quad \forall a_i \in A.$$

Thus we obtain a decomposition of cocyclic modules

$$C_{\mathcal{H}}^n(A) \simeq \prod_{-\infty}^{+\infty} C_{\theta, m}^n(A),$$

where $C_{\theta, m}^n(A) = \{f : A^{\otimes(n+1)} \rightarrow k, f(\theta a_0, \dots, \theta a_n) = f(a_0, \dots, a_n)\}$. The coface and cyclic operators of $\{C_{\theta, m}^n(A)\}_{n \geq 0}$ are given by

$$\begin{aligned} (\delta_i f)(a_0, \dots, a_n) &= f(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}), \quad 0 \leq i \leq n, \\ (\delta_{n+1} f)(a_0, \dots, a_n) &= f((\theta^m a_{n+1}) a_0, a_1, \dots, a_n), \\ (t f)(a_0, \dots, a_n) &= f(\theta^m a_n, a_0, \dots, a_{n-1}). \end{aligned}$$

For $m = 1$, the cocyclic module $\{C_{\theta, 1}^n(A)\}_{n \geq 0}$ is exactly the cocyclic module in [12], used to define the " θ -twisted cyclic cohomology" of A .

4 Connection with the cocyclic module $\text{Hom}_k((A \rtimes \mathcal{H})^{\natural}, k)$

In this section we define a cocyclic map between the cocyclic module $C_{\mathcal{H}}^{\natural}(A)$ of equivariant cochains on A , introduced in the previous section and the cocyclic module $\text{Hom}_k((A \rtimes \mathcal{H})^{\natural}, k)$, associated with the crossed product algebra $A \rtimes \mathcal{H}$.

Define a k -linear map

$$\varphi_n : C_{\mathcal{H}}^n(A) \rightarrow \text{Hom}_k((A \rtimes \mathcal{H})^{\otimes(n+1)}, k),$$

by

$$\varphi_n f(a_0 \otimes g_0, \dots, a_n \otimes g_n) :=$$

$$\begin{aligned} f(S^{-1}(g_0^{(0)} g_1^{(1)} g_2^{(2)} \cdots g_n^{(n)}) \cdot a_0, S^{-1}(g_1^{(0)} g_2^{(1)} \cdots g_n^{(n-1)}) \cdot a_1, \\ \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1}, S^{-1}(g_n^{(0)}) \cdot a_n) (g_0^{(1)} g_1^{(2)} \cdots g_n^{(n)}). \end{aligned}$$

Let $\varphi = \{\varphi_n\}_{n \geq 0}$. Now we can state our first main result in this section.

Theorem 4.1. φ defines a cocyclic map between cocyclic modules $C_{\mathcal{H}}^{\natural}(A)$ and $\text{Hom}_k((A \rtimes \mathcal{H})^{\natural}, k)$.

Proof. First we show that φ commutes with cyclic operators. We have

$$\begin{aligned}
& (\varphi_n T_n f)(a_0 \otimes g_0, \dots, a_n \otimes g_n) = \\
& T_n f(S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_n^{(n)}) \cdot a_0, S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_n^{(n-1)}) \cdot a_1, \\
& \quad \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1}, S^{-1}(g_n^{(0)}) \cdot a_n)(g_0^{(1)} g_1^{(2)} \dots g_n^{(n+1)}) \\
& = f((S^{-1}(g_0^{(1)} g_1^{(2)} \dots g_n^{(n+1)}) S^{-1}(g_n^{(0)})) \cdot a_n, S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_n^{(n)}) \cdot a_0, \\
& \quad \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1})(g_0^{(2)} g_1^{(3)} \dots g_n^{(n+2)}) \\
& = f((S^{-1}(g_n^{(n+1)}) S^{-1}(g_n^{(0)} g_0^{(1)} \dots g_{n-1}^{(n)})) \cdot a_n, (S^{-1}(g_n^{(n)}) S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)})) \cdot a_0, \\
& \quad \dots, (S^{-1}(g_n^{(1)}) S^{-1}(g_{n-1}^{(0)})) \cdot a_{n-1})(g_0^{(2)} g_1^{(3)} \dots g_n^{(n+2)}) \\
& = S^{-1}(g_n^{(1)}) \cdot f(S^{-1}(g_n^{(0)} g_0^{(1)} \dots g_{n-1}^{(n)}) \cdot a_n, S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)}) \cdot a_0, \\
& \quad \dots, S^{-1}(g_{n-1}^{(0)}) \cdot a_{n-1})(g_0^{(2)} g_1^{(3)} \dots g_n^{(n+2)}) \\
& = f(S^{-1}(g_n^{(0)} g_0^{(1)} \dots g_{n-1}^{(n)}) \cdot a_n, S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)}) \cdot a_0, \\
& \quad \dots, S^{-1}(g_{n-1}^{(0)}) \cdot a_{n-1})(S^{-2}(g_n^{(1)}) \cdot (g_0^{(2)} g_1^{(3)} \dots g_n^{(n+2)})).
\end{aligned}$$

Since

$$\begin{aligned}
& S^{-2}(g_n^{(1)}) \cdot (g_0^{(2)} g_1^{(3)} \dots g_n^{(n+2)}) = g_n^{(1)} g_0^{(2)} g_1^{(3)} \dots g_{n-1}^{(n+1)} g_n^{(3)} S^{-1}(g_n^{(2)}) \\
& = \epsilon(g_n^{(2)}) g_n^{(1)} g_0^{(2)} g_1^{(3)} \dots g_{n-1}^{(n+1)} = g_n^{(1)} g_0^{(2)} g_1^{(3)} \dots g_{n-1}^{(n+1)},
\end{aligned}$$

we obtain

$$\begin{aligned}
& (\varphi_n T_n f)(a_0 \otimes g_0, \dots, a_n \otimes g_n) = \\
& = f(S^{-1}(g_n^{(0)} g_0^{(1)} g_1^{(2)} \dots g_{n-1}^{(n)}) \cdot a_n, S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)}) \cdot a_0, \\
& \quad S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_{n-1}^{(n-2)}) \cdot a_1, \dots, S^{-1}(g_{n-1}^{(0)}) \cdot a_{n-1})(g_n^{(1)} g_0^{(2)} \dots g_{n-1}^{(n+1)}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
& (\tau_n \varphi_n f)(a_0 \otimes g_0, \dots, a_n \otimes g_n) = \varphi_n f(a_n \otimes g_n, a_0 \otimes g_0, \dots, a_{n-1} \otimes g_{n-1}) \\
& = f(S^{-1}(g_n^{(0)} g_0^{(1)} g_1^{(2)} \dots g_{n-1}^{(n)}) \cdot a_n, S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)}) \cdot a_0, \\
& \quad \dots, S^{-1}(g_{n-2}^{(0)} g_{n-1}^{(1)}) \cdot a_{n-2}, S^{-1}(g_{n-1}^{(0)}) \cdot a_{n-1})(g_n^{(1)} g_0^{(2)} \dots g_{n-1}^{(n+1)}).
\end{aligned}$$

Thus, φ commutes with cyclic operators.

Next we show that φ commutes with coface operators, i.e., $\partial^i \varphi_{n-1} = \varphi_n \partial^i$. We check this only for $i = n$ and leave the rest to the reader. We have

$$\begin{aligned}
& (\partial^n \varphi_{n-1} f)(a_0 \otimes g_0, \dots, a_n \otimes g_n) \\
&= (\varphi_{n-1} f)((a_n \otimes g_n)(a_0 \otimes g_0), a_1 \otimes g_1, \dots, a_{n-1} \otimes g_{n-1}) \\
&= (\varphi_{n-1} f)(a_n(g_n^{(0)}) \cdot a_0 \otimes g_n^{(1)} g_0, a_1 \otimes g_1, \dots, a_{n-1} \otimes g_{n-1}) \\
&= f(S^{-1}(g_n^{(1)} g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)}) \cdot (a_n(g_n^{(0)}) \cdot a_0)), S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_{n-1}^{(n-2)}) \cdot a_1, \\
&\quad \dots, S^{-1}(g_{n-1}^{(0)}) \cdot a_{n-1})(g_n^{(2)} g_0^{(1)} \dots g_{n-1}^{(n)}),
\end{aligned}$$

and

$$\begin{aligned}
& (\varphi_n \partial^n f)(a_0 \otimes g_0, \dots, a_n \otimes g_n) = \\
& \partial^n f(S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_n^{(n)}) \cdot a_0, S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_n^{(n-1)}) \cdot a_1, \\
& \quad \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1}, S^{-1}(g_n^{(0)}) \cdot a_n)(g_0^{(1)} g_1^{(2)} \dots g_n^{(n+1)}) \\
&= f(((S^{-1}(g_0^{(1)} g_1^{(2)} \dots g_n^{(n+1)}) S^{-1}(g_n^{(0)})) \cdot a_n)(S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_n^{(n)}) \cdot a_0), \\
& \quad S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_n^{(n-1)}) \cdot a_1, \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1})(g_0^{(2)} g_1^{(3)} \dots g_n^{(n+2)}) \\
&= f((S^{-1}(g_n^{(0)} g_0^{(1)} g_1^{(2)} \dots g_n^{(n+1)}) \cdot a_n)(S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_n^{(n)}) \cdot a_0), \\
& \quad S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_n^{(n-1)}) \cdot a_1, \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1})(g_0^{(2)} g_1^{(3)} \dots g_n^{(n+2)}) \\
&= f(S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_n^{(n)}) \cdot (S^{-1}(g_n^{(0)}) \cdot a_n) a_0, S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_n^{(n-1)}) \cdot a_1, \\
& \quad \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1})(g_0^{(1)} g_1^{(2)} \dots g_n^{(n+1)}) \\
&= f((S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)} g_n^{(n+1)}) S^{-1}(g_n^{(1)}))(a_n(g_n^{(0)}) \cdot a_0), \\
& \quad S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_{n-1}^{(n-2)} g_n^{(n)}) \cdot a_1, \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(2)}) \cdot a_{n-1})(g_0^{(1)} g_1^{(2)} \dots g_{n-1}^{(n)} g_n^{(n+2)}) \\
&= S^{-1}(g_n^{(2)}) \cdot f(S^{-1}(g_n^{(1)} g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)}) \cdot (a_n(g_n^{(0)}) \cdot a_0), \\
& \quad S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_{n-1}^{(n-2)}) \cdot a_1, \dots, S^{-1}(g_{n-1}^{(0)}) \cdot a_{n-1})(g_0^{(1)} g_1^{(2)} \dots g_{n-1}^{(n)} g_n^{(3)}) \\
&= f(S^{-1}(g_n^{(1)} g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)}) \cdot (a_n(g_n^{(0)}) \cdot a_0), S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_{n-1}^{(n-2)}) \cdot a_1, \\
& \quad \dots, S^{-1}(g_{n-1}^{(0)}) \cdot a_{n-1})(g_n^{(2)} g_0^{(1)} g_1^{(2)} \dots g_{n-1}^{(n)} g_n^{(4)} S^{-1}(g_n^{(3)})).
\end{aligned}$$

Since $g_n^{(4)} S^{-1}(g_n^{(3)}) = \epsilon(g_n^{(3)})$ and $\epsilon(g_n^{(3)}) g_n^{(2)} = g_n^{(2)}$, the result is

$$\begin{aligned} & (\varphi_n \partial^n f)(a_0 \otimes g_0, \dots, a_n \otimes g_n) \\ &= f(S^{-1}(g_n^{(1)} g_0^{(0)} g_1^{(1)} \dots g_{n-1}^{(n-1)}) \cdot (a_n(g_n^{(0)} \cdot a_0)), S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_{n-1}^{(n-2)}) \cdot a_1, \\ & \quad \dots, S^{-1}(g_{n-1}^{(0)} \cdot a_{n-1})(g_n^{(2)} g_0^{(1)} g_1^{(2)} \dots g_{n-1}^{(n)}). \end{aligned}$$

The proof of compatibility of φ with codegeneracies is similar and we leave it to the reader. The theorem is proved. \square

Corollary 4.1. φ induces natural maps between Hochschild, cyclic and periodic cyclic cohomologies of $C_{\mathcal{H}}^{\natural}(A)$ and $\text{Hom}_k((A \rtimes \mathcal{H})^{\natural}, k)$:

$$\begin{aligned} HH_{\mathcal{H}}^*(A) & \xrightarrow{\varphi_H} HH^*(A \rtimes \mathcal{H}), \\ HC_{\mathcal{H}}^*(A) & \xrightarrow{\varphi_C} HC^*(A \rtimes \mathcal{H}), \\ HP_{\mathcal{H}}^*(A) & \xrightarrow{\varphi_P} HP^*(A \rtimes \mathcal{H}). \end{aligned}$$

Theorem 4.2. There is a spectral sequence that converges to the cyclic cohomology of $A \rtimes \mathcal{H}$. The E_2 -term of this spectral sequence is given by

$$E_2^{p,q} = H^p(\mathcal{H}, C_{\mathcal{H}}^q(A)).$$

Proof. We construct a cocylindrical module $X = \{X_{p,q}\}_{p,q \geq 0}$ and show that the diagonal $\Delta(X)$ of X is isomorphic to the cocyclic module $\text{Hom}((A \rtimes \mathcal{H})^{\natural}, k)$. We can then apply the cyclic Eilenberg-Zilber theorem to derive our spectral sequence. Let

$$X_{p,q} = \text{Hom}(A^{\otimes(p+1)} \otimes \mathcal{H}^{\otimes(q+1)}, k).$$

We define the horizontal and vertical cosimplicial and cyclic operators by

$$\begin{aligned} \tau_{p,q} f(a_0, \dots, a_p)(g_0, \dots, g_q) &= f(S^{-1}(g_0^{(0)} \dots g_q^{(0)}) \cdot a_p, a_0, \dots, a_{p-1})(g_0^{(1)}, \dots, g_q^{(1)}), \\ \partial_{p,q}^i f(a_0, \dots, a_p)(g_0, \dots, g_q) &= f(a_0, \dots, a_i a_{i+1}, \dots, a_p)(g_0, \dots, g_q), \quad 0 \leq i \leq p-1, \\ \partial_{p,q}^p f(a_0, \dots, a_p)(g_0, \dots, g_q) &= f((S^{-1}(g_0^{(0)} \dots g_q^{(0)}) \cdot a_p) a_0, \dots, a_{p-1})(g_0^{(1)}, \dots, g_q^{(1)}), \\ \sigma_{p,q}^i f(a_0, \dots, a_p)(g_0, \dots, g_q) &= f(a_0, \dots, a_i, 1, a_{i+1}, \dots, a_p)(g_0, \dots, g_q), \quad 0 \leq i \leq p, \\ \bar{\tau}_{p,q} f(a_0, \dots, a_p)(g_0, \dots, g_q) &= f(g_q^{(0)} \cdot (a_0, \dots, a_p))(g_q^{(1)}, g_0, \dots, g_{q-1}), \\ \bar{\partial}_{p,q}^i f(a_0, \dots, a_p)(g_0, \dots, g_q) &= f(a_0, \dots, a_p)(g_0, \dots, g_i g_{i+1}, \dots, g_q), \quad 0 \leq i \leq q-1, \\ \bar{\partial}_{p,q}^q f(a_0, \dots, a_p)(g_0, \dots, g_q) &= f(g_q^{(0)} \cdot (a_0, \dots, a_p))(g_q^{(1)} g_0, \dots, g_{q-1}), \\ \bar{\sigma}_{p,q}^i f(a_0, \dots, a_p)(g_0, \dots, g_q) &= f(a_0, \dots, a_p)(g_0, \dots, g_i, 1, g_{i+1}, \dots, g_q), \quad 0 \leq i \leq q. \end{aligned}$$

One can check that $\{X_{p,q}\}_{p,q \geq 0}$ is a cocylindrical module. The proof is very long, but is totally similar to the proof of Theorem 3.1 in [1] and is left to the reader.

Next we show that the diagonal of X , $\Delta(X)$, is isomorphic with the cocyclic module $Hom((A \rtimes \mathcal{H})^{\natural}, k)$. To this end, we define the maps $\varphi = \{\varphi_n\}_{n \geq 0}$ and $\psi = \{\psi_n\}_{n \geq 0}$ by

$$\varphi_n : \Delta^n(X) \rightarrow Hom((A \rtimes \mathcal{H})^{\otimes(n+1)}, k), \quad \psi_n : Hom((A \rtimes \mathcal{H})^{\otimes(n+1)}, k) \rightarrow \Delta^n(X),$$

$$\begin{aligned} \varphi_n f(a_0 \otimes g_0, \dots, a_n \otimes g_n) = \\ f(S^{-1}(g_0^{(0)} g_1^{(1)} g_2^{(2)} \cdots g_n^{(n)}) \cdot a_0, S^{-1}(g_1^{(0)} g_2^{(1)} \cdots g_n^{(n-1)}) \cdot a_1, \\ \dots, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1}, S^{-1}(g_n^{(0)}) \cdot a_n)(g_0^{(1)}, g_1^{(2)}, \dots, g_n^{(n)}), \end{aligned}$$

$$\begin{aligned} \psi_n f(a_0, \dots, a_n)(g_0, \dots, g_n) = \\ f((g_0^{(0)} \cdots g_{n-1}^{(0)} g_n^{(0)}) \cdot a_0 \otimes g_0^{(1)}, (g_1^{(1)} \cdots g_{n-1}^{(1)} g_n^{(1)}) \cdot a_1 \otimes g_1^{(2)}, \dots, g_n^{(n)} \cdot a_n \otimes g_n^{(n+1)}). \end{aligned}$$

By a rather long computation one can verify that ϕ is a morphism of cocyclic modules and $\phi \circ \psi = \psi \circ \phi = id$. Now, we can apply the generalized cyclic Eilenberg-Zilber theorem to derive our spectral sequence. Again the argument is similar to that used in [1, 8] and hence omitted. \square

Corollary 4.2. *Assume \mathcal{H} is semisimple. Then we have an isomorphism of cyclic cohomology groups $HC^\bullet(A \rtimes \mathcal{H}) \simeq HC^\bullet_{\mathcal{H}}(A)$.*

Remark. One can develop a similar theory for Hopf comodule algebras and prove the analogues of Theorem 4.2 and Corollary 4.2, for cosemisimple Hopf algebras. It is known that compact quantum groups in the sense of Woronowicz are cosemisimple [11].

We should also mention that some of our constructions and definitions in Sections 3 and 4, once appropriately dualized, reduce to those considered by Nistor in [16]. In particular, let G be a compact Lie group acting smoothly on a complete locally convex algebra A , and let $\mathcal{H} = Rep(G) \subset C^\infty(G)$ be the Hopf algebra of representable functions on G [11]. Then, the dual of Corollary 4.2 reduces to Proposition 3.4 in [16].

5 Equivariant K-Theory

In this section we define the equivariant K_0 -theory of Yetter-Drinfeld algebras and show that there exists a pairing, generalizing Connes' Chern character [6], between this theory and the equivariant cyclic cohomology defined in Section 3. One can perhaps define an equivariant K_0 -theory for any Hopf module algebra using finitely generated projective modules endowed with a compatible action of the Hopf algebra. It is however not clear how to define a Chern character map in this setting. Our approach, based on idempotents, however, naturally led us to a special class of \mathcal{H} -algebras, namely the Yetter-Drinfeld \mathcal{H} -algebras.

Let \mathcal{H} be a Hopf algebra with a bijective antipode. By a *Yetter-Drinfeld \mathcal{H} -algebra* [11] we mean an algebra A that satisfies the following conditions:

- 1) A is a left \mathcal{H} -algebra,
- 2) A is a right \mathcal{H}^{op} -comodule algebra, i.e., the coaction $\rho : A \rightarrow A \otimes \mathcal{H}$, satisfies

$$\rho(ab) = a_{<0>} b_{<0>} \otimes b_{<1>} a_{<1>},$$

where $\rho(a) = a_{<0>} \otimes a_{<1>} \in A \otimes \mathcal{H}$, denotes the coaction.

3) Conditions 1) and 2) are compatible in the sense that they satisfy the Yetter-Drinfeld condition

$$(h^{(1)} \cdot a)_{<0>} \otimes (h^{(1)} \cdot a)_{<1>} h^{(0)} = h^{(0)} \cdot a_{<0>} \otimes h^{(1)} a_{<1>}, \quad h \in \mathcal{H}, a \in A. \quad (7)$$

We denote the class of Yetter-Drinfeld algebras of the above type by ${}_{\mathcal{H}}\mathcal{YD}^{\mathcal{H}}$. It is easily checked that if \mathcal{H} is cocommutative, then any left \mathcal{H} -algebra is a Yetter-Drinfeld algebra with a coaction defined by $a \rightarrow a \otimes 1$.

Lemma 5.1. *Given any left \mathcal{H} -algebra B and a Yetter-Drinfeld algebra A , then $A \otimes B$ with diagonal action and the following multiplication is an \mathcal{H} -algebra:*

$$(a \otimes b)(c \otimes d) = ac_{<0>} \otimes (c_{<1>} \cdot b)d. \quad (8)$$

The \mathcal{H} -action on $A \otimes B$ is diagonal, i.e.,

$$h \cdot (a \otimes b) = h^{(0)} \cdot a \otimes h^{(1)} \cdot b.$$

Proof. It is not difficult to see that (8) defines an associative product on $A \otimes B$. We check the \mathcal{H} -module algebra condition.

$$\begin{aligned} & (h^{(0)} \cdot (a \otimes b))(h^{(1)} \cdot (c \otimes d)) = (h^{(0)} \cdot a \otimes h^{(1)} \cdot b)(h^{(2)} \cdot c \otimes h^{(3)} \cdot d) \\ &= (h^{(0)} \cdot a)(h^{(2)} \cdot c)_{<0>} \otimes ((h^{(2)} \cdot c)_{<1>} \cdot (h^{(1)} \cdot b))(h^{(3)} \cdot d) \\ &= (h^{(0)} \cdot a)(h^{(2)} \cdot c)_{<0>} \otimes ((h^{(2)} \cdot c)_{<1>} h^{(1)} \cdot b)(h^{(3)} \cdot d) \\ &= (h^{(0)} \cdot a)(h^{(1)} \cdot c_{<0>}) \otimes ((h^{(2)} c_{<1>}) \cdot b)(h^{(3)} \cdot d) \\ &= (h^{(0)} \cdot a)(h^{(1)} \cdot c_{<0>}) \otimes (h^{(2)} \cdot (c_{<1>} \cdot b))(h^{(3)} \cdot d) \\ &= h \cdot (ac_{<0>} \otimes (c_{<1>} \cdot b)d) = h \cdot ((a \otimes b)(c \otimes d)). \end{aligned}$$

□

Lemma 5.2. *Let \mathcal{H} be a Hopf algebra with an invertible antipode and A a Yetter-Drinfeld algebra. Then we have:*

$$\rho(h \cdot a) = h^{(1)} \cdot a_{<0>} \otimes h^{(2)} a_{<1>} S^{-1}(h^{(0)}). \quad (9)$$

Proof. By (7) we can see that

$$\begin{aligned} h^{(1)} \cdot a_{<0>} \otimes h^{(2)} a_{<1>} S^{-1}(h^{(0)}) &= (h^{(2)} \cdot a_{<1>})_{<0>} \otimes (h^{(2)} \cdot a_{<1>})_{<1>} h^{(1)} S^{-1}(h^{(0)}) \\ &= (h \cdot a)_{<0>} \otimes (h \cdot a)_{<1>}. \end{aligned}$$

□

Conversely, one can check that condition (9) implies the Yetter-Drinfeld condition (7).

Now let V be a representation of \mathcal{H} , i.e., V is a left \mathcal{H} -module with structure map $r : \mathcal{H} \rightarrow \text{End}(V)$. Then $B = \text{End}(V)$, with conjugation action

$$h \cdot f = r(h^{(0)}) \circ f \circ r(S(h^{(1)})),$$

is an \mathcal{H} -algebra. Let $A \in {}_{\mathcal{H}}\mathcal{YD}^{\mathcal{H}}$. Then by Lemma 5.1, $A \otimes \text{End}(V)$ is an \mathcal{H} -algebra with diagonal action and twisted multiplication i.e.,

$$(a \otimes u)(c \otimes v) = ac_{<0>} \otimes (c_{<1>} \cdot u)v, \quad h \cdot (a \otimes u) = h^{(0)} \cdot a \otimes h^{(1)} \cdot u. \quad (10)$$

To simplify the notation, we denote the image of h under r by h itself.

Let A be an \mathcal{H} -algebra. We say that $b \in A$ is an \mathcal{H} -invariant element if, for every $h \in \mathcal{H}$, $h \cdot b = \epsilon(h)b$. For a Yetter-Drinfeld algebra A we define $P_{\mathcal{H}}(A)$ to be the set of all \mathcal{H} -invariant idempotents in all of the algebras $A \otimes \text{End}(V)$, where V is a finite dimensional representation of \mathcal{H} . For $e, e' \in P_{\mathcal{H}}(A)$, $e \in A \otimes \text{End}(V)$ and $e' \in A \otimes \text{End}(W)$, we define their sum $e_1 \oplus e_2$ as $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in A \otimes \text{End}(V \oplus W)$.

Two \mathcal{H} -invariant idempotents $e \in A \otimes \text{End}(V)$ and $e' \in A \otimes \text{End}(W)$ are called Murray-von Neumann equivalent if there exist \mathcal{H} -invariant elements $\gamma_1 \in A \otimes \text{Hom}_k(V, W)$ and $\gamma_2 \in A \otimes \text{Hom}_k(W, V)$ such that $\gamma_2 \gamma_1 = e$ and $\gamma_1 \gamma_2 = e'$. Let $S_{\mathcal{H}}(A)$ denote the set of equivalence classes of $P_{\mathcal{H}}(A)$ under the Murray-von Neumann equivalence relation. It is clear that $S_{\mathcal{H}}(A)$ is an abelian semigroup under the direct sum of idempotents.

If there exists an \mathcal{H} -invariant invertible element $\gamma \in \text{Hom}_k(V, W) \otimes A$ such that $\gamma e \gamma^{-1} = e'$, we say that e and e' are *similar* and we write $e \sim e'$.

The proof of the following theorem is similar to the case of group actions (Prop. 2.4.11 in [17]),

Proposition 5.1. *$S_{\mathcal{H}}(A)$ is equal to the set of equivalence classes in $P_{\mathcal{H}}(A)$ for the equivalence relation generated by similarity and the relation $p \sim p \oplus 0$.*

We define the equivariant K_0 -theory of a Yetter-Drinfeld algebra A over a Hopf algebra \mathcal{H} , denoted by $K_0^{\mathcal{H}}(A)$, as the Grothendieck group of $S_{\mathcal{H}}(A)$.

Let $(A^{\mathcal{H}})^{\times}$ be the group of invertible \mathcal{H} -invariant elements of A . Any element $b \in (A^{\mathcal{H}})^{\times}$ acts by conjugation on $C_{\mathcal{H}}^n(A)$ by the formula

$$\alpha_b f(a_0, a_1, \dots, a_n)(g) = f(ba_0b^{-1}, ba_1b^{-1}, \dots, ba_nb^{-1})(g),$$

and any \mathcal{H} -invariant element $b \in A^{\mathcal{H}}$, acts by inner derivations by the formula

$$\delta_b f(a_0, a_1, \dots, a_n)(g) = \sum_{i \geq 0} f(a_0, \dots, a_{i-1}, [b, a_i], a_{i+1}, \dots, a_n)(g).$$

Lemma 5.3. α_b and δ_b are equivariant cocyclic maps.

Lemma 5.4. α_b induces the identity map and δ_b induces the zero map on $HH_{\mathcal{H}}^*(A)$, $HC_{\mathcal{H}}^*(A)$ and $HP_{\mathcal{H}}^*(A)$.

Proof. We see that the maps $\theta^i : C_{\mathcal{H}}^{n+1}(A) \rightarrow C_{\mathcal{H}}^n(A)$, $0 \leq i \leq n$, where

$$\theta^i f(a_0, a_1, \dots, a_n)(g) = f(a_0 b^{-1}, b a_1 b^{-1}, \dots, b a_i b^{-1}, b, a_{i+1}, \dots, a_n)(g),$$

define a presimplicial homotopy between $\theta^n \partial^{n+1} = \alpha_b$ and $\theta^0 \partial^0 = id$, so that α_b is an isomorphism on Hochschild cohomology and hence on (periodic)cyclic cohomology. In the case of δ_b we see that since

$$\partial^{n+1} f(a_0, a_1, \dots, a_n, b)(g) = f(S(g^{(0)}) \cdot b, a_0, \dots, a_n)(g^{(1)}) = f(b, a_0, \dots, a_n)(g),$$

the map

$$\varrho f(a_0, a_1, \dots, a_n)(g) = \sum_{0 \leq i \leq n} (-1)^i f(a_0, \dots, a_i, b, a_{i+1}, \dots, a_n)(g),$$

defines the homotopy $b\varrho + \varrho b = -\delta_b$ between δ_b and 0. So, δ_b is 0 on Hochschild cohomology and since $[b, 1] = 0$, in the normalized form $\varrho B + B\varrho = 0$ and ϱ defines a homotopy between δ_b and 0. Thus, δ_b is 0 on cyclic and periodic cyclic cohomology. \square

Let V be a finite dimensional \mathcal{H} -module. We construct the *generalized trace map* Ψ^n between $C_{\mathcal{H}}^n(A)$ and $C_{\mathcal{H}}^n(A \otimes \text{End}(V))$. We define

$$\Psi^n : C_{\mathcal{H}}^n(A) \rightarrow C_{\mathcal{H}}^n(A \otimes \text{End}(V)),$$

$$\begin{aligned} \Psi^n f(a_0 \otimes u_0, \dots, a_n \otimes u_n)(g) &= f(a_{0<0>}, a_{1<0>}, \dots, a_{n<0>})(g^{(1)}) \\ &\quad \text{tr}(S(a_{0<1>})u_0 S(a_{1<1>})u_1 \dots S(a_{n<1>})u_n g^{(0)}). \end{aligned} \quad (11)$$

Proposition 5.2. Ψ is an equivariant cocyclic map.

Proof. We first prove that Ψ^n is equivariant, i.e., if f is an equivariant cochain then $h \cdot \Psi^n f(g) = \Psi^n f(S^{-1}(h) \cdot g) = \Psi^n f(S(h^{(1)})gh^{(0)})$:

$$\begin{aligned} &h \cdot (\Psi^n f)(a_0 \otimes u_0, \dots, a_n \otimes u_n)(g) \\ &= \Psi^n f(h^{(0)} \cdot a_0 \otimes h^{(1)} \cdot u_0, \dots, h^{(2n)} \cdot a_n \otimes h^{(2n+1)} \cdot u_n)(g) \\ &= f(h^{(1)} \cdot a_{0<0>}, h^{(6)} \cdot a_{1<0>}, \dots, h^{(5n+1)} \cdot a_{n<0>})(g^{(1)}) \\ &\quad \text{tr}(S(h^{(2)} a_{0<1>} S^{-1}(h^{(0)}))(h^{(3)} u_0 S(h^{(4)})) S(h^{(7)} a_{1<1>} S^{-1}(h^{(5)}))(h^{(8)} u_1 S(h^{(9)})) \dots \\ &\quad \quad \quad S(h^{(5n+2)} a_{n<1>} S^{-1}(h^{(5n)}))(h^{(5n+3)} u_n S(h^{(5n+4)})) g^{(0)}) \\ &= f(h^{(1)} \cdot a_{0<0>}, h^{(2)} \cdot a_{1<0>}, \dots, h^{(n+1)} \cdot a_{n<0>})(g^{(1)}) \\ &\quad \text{tr}(h^{(0)} S(a_{0<1>})u_0 S(a_{1<1>})u_1 \dots S(a_{n<1>})u_n S(h^{(n+2)})g^{(0)}) \end{aligned}$$

$$\begin{aligned}
&= h^{(1)} \cdot f(a_{0<0>}, a_{1<0>}, \dots, a_{n<0>})(g^{(1)}) \\
&\quad \text{tr}(S(a_{0<1>})u_0 S(a_{1<1>})u_1 \dots S(a_{n<1>})u_n S(h^{(2)})g^{(0)}h^{(0)}) \\
&= f(a_0, a_1, \dots, a_n)(S^{-1}(h^{(1)}) \cdot g) \\
&\quad \text{tr}(S(a_{0<1>})u_0 S(a_{1<1>})u_1 \dots S(a_{n<1>})u_n S(h^{(2)})g^{(0)}h^{(0)}) \\
&= (\Psi^n f)(a_0 \otimes v_0, \dots, a_n \otimes v_n)(S(h^{(1)})gh^{(0)}).
\end{aligned}$$

Next we check that Ψ is a cyclic map. First we check that Ψ^n commutes with the cyclic operators:

$$\begin{aligned}
&(T_n \Psi^n f)(a_0 \otimes u_0, a_1 \otimes u_1, \dots, a_n \otimes u_n)(g) \\
&= \Psi^n f(S^{-1}(g^{(1)}) \cdot a_n \otimes S^{-1}(g^{(0)}) \cdot u_n, a_0 \otimes u_0, \dots, a_{n-1} \otimes u_{n-1})(g^{(2)}) \\
&= f(S^{-1}(g^{(3)}) \cdot a_n, a_0, \dots, a_{n-1})(g^{(6)}) \\
&\quad \text{tr}(S(S^{-1}(g^{(2)})a_{n<1>}S^{-2}(g^{(4)}))S^{-1}(g^{(1)})u_n g^{(0)}S(a_{0<1>})u_0 \dots S(a_{n-1<1>})u_{n-1}g^{(5)}) \\
&= f(S^{-1}(g^{(1)}) \cdot a_n, a_0, \dots, a_{n-1})(g^{(2)})\text{tr}(S(a_{0<1>})u_0 S(a_{1<1>})u_1 \dots S(a_{n<1>})u_n g^{(0)}) \\
&= (\Psi^n T_n f)(a_0 \otimes u_0, a_1 \otimes u_1, \dots, a_n \otimes u_n)(g).
\end{aligned}$$

Now since A is a right \mathcal{H}^{op} -comodule algebra, we have $\rho(ab) = a_{<0>}b_{<0>} \otimes b_{<1>}a_{<1>}$, and since

$$(a_i \otimes u_i)(a_{i+1} \otimes u_{i+1}) = a_i a_{i+1<0>} \otimes (a_{i+1<1>} \cdot u_i)u_{i+1},$$

we can see that

$$\begin{aligned}
S((a_i a_{i+1<0>})_{<1>})((a_{i+1<1>} \cdot u_i)u_{i+1}) &= S(a_{i+1<1>}a_{i<1>})(a_{i+1<2>}u_i S(a_{i+1<3>})u_{i+1}) \\
&= S(a_{i<1>})u_i S(a_{i+1<1>})u_{i+1}.
\end{aligned}$$

Therefore Ψ^n commutes with coface operators ∂^i , $0 \leq i < n$. Since $T_n \partial^0 = \partial^n$, Ψ^n commutes with all coface operators and it is easy to check that it also commutes with codegeneracy operators. This proves the proposition. \square

Corollary 5.1. (*Morita invariance*) *Let V be a trivial \mathcal{H} -module. Then $HC_{\mathcal{H}}^*(A) \simeq HC_{\mathcal{H}}^*(A \otimes \text{End}(V))$.*

Proof. As is shown in [9], Theorem 6, Morita invariance is a formal consequence of two facts: inner automorphisms induce the identity map on cohomology and a generalized trace map exists. In our case, these are established in Lemma 5.4 and Proposition 5.2. \square

Now we state the main result of this section. We define $R(\mathcal{H})$ to be the space of all invariant functions from \mathcal{H} to the ground field k . So $f \in R(\mathcal{H})$ iff $f(h \cdot g) = f(S^2(h^{(0)})gS(h^{(1)})) = \epsilon(h)f(g)$.

Theorem 5.1. *For each $n \geq 0$ there exists a bilinear pairing $K_0^{\mathcal{H}}(A) \times HC_{\mathcal{H}}^{2n}(A) \rightarrow R(\mathcal{H})$, defined by*

$$\langle [e], (f) \rangle(g) = \Psi^{2n} f(e, e, \dots, e)(g).$$

We also have a pairing $K_0^{\mathcal{H}}(A) \times HP_{\mathcal{H}}^0(A) \rightarrow R(\mathcal{H})$, defined by

$$\langle [e], (f) \rangle(g) = \Psi f_0(e)(g) + \sum_n (-1)^n \frac{(2n)!}{n!} \Psi f_{2n}(e - \frac{1}{2}, e, \dots, e)(g),$$

where in the last pairing $(f) = (f_{2n})_{0 \leq n \leq m}$ is an equivariant even periodic cyclic cocycle in the normalized equivariant (b, B) -bicomplex of A .

Proof. To check that the first pairing is well defined, let f be a coboundary in $C_{\mathcal{H}}^{2n}(A)$. Then $\Psi^{2n} f = b\psi^{2n-1}$ is also a coboundary and we see that

$$\begin{aligned} \Psi^{2n} f(e, e, \dots, e)(g) &= b\psi^{2n-1}(e, e, \dots, e)(g) = \sum_{0 \leq i \leq n} (-1)^i \partial^i \psi^{2n-1}(e, e, \dots, e)(g) \\ &= \psi^{2n-1}(e^2, e, \dots, e)(g) - \psi^{2n-1}(e, e^2, \dots, e)(g) + \dots + (-1)^{2n-2} \psi^{2n-1}(e, e, \dots, e^2)(g) \\ &\quad + (-1)^{2n-1} \psi^{2n-1}((S^{-1}(g^{(0)}) \cdot e)e, e, \dots, e)(g^{(1)}) = 0, \end{aligned}$$

since e is \mathcal{H} -invariant and therefore $S^{-1}(g^{(0)}) \cdot e = \epsilon(g^{(0)})e$.

Let $[e] = [e']$ in $K_0^{\mathcal{H}}(A)$, where $e \in A \otimes \text{End}(V)$ and $e' \in A \otimes \text{End}(W)$. Then, by Proposition 5.1, $e \sim e \oplus 0 \sim 0 \oplus e' \sim e'$, so there exists an \mathcal{H} -invariant invertible element $\gamma \in A \otimes \text{End}(V \oplus W)$ such that $e' \oplus 0 = \gamma(0 \oplus e)\gamma^{-1}$. Then, by Lemma 5.4, we have

$$\Psi^{2n} f(e', \dots, e')(g) = \Psi^{2n} f(\gamma e \gamma^{-1}, \dots, \gamma e \gamma^{-1})(g) = \Psi^{2n} f(e, \dots, e)(g).$$

Also, since f is equivariant, $\langle [e], (f) \rangle \in R(\mathcal{H})$. This finishes the proof of the first part. The proof of the second part is also similar to the non equivariant case as in [13] and is left to the reader. \square

6 Examples

In this section we first give some examples of Yetter-Drinfeld module algebras and show that for Yetter-Drinfeld module algebras naturally defined by R -matrix of (co)quasitriangular Hopf algebras the two definitions of $K_0^{\mathcal{H}}$ as given in this paper and in [14] coincide. We then generalize this result and show that the two definitions coincide for all Yetter-Drinfeld module algebras. Finally, we show that the quantum analogue of the Dirac monopole line bundle over the quantum sphere S_q^2 defines an element of $U_q(su_2)$ -equivariant K -theory of S_q^2 .

Let A be an \mathcal{H} -module algebra. Then one can check that $A \otimes \mathcal{H}$ with the following structure is a Yetter-Drinfeld module algebra over \mathcal{H} :

$$(a \otimes h)(b \otimes g) = (ab \otimes hg),$$

$$g \cdot (a \otimes h) = g^{(1)} \otimes g^{(0)} h S(g^{(2)}), \quad \rho(a \otimes h) = a \otimes h^{(1)} \otimes S^{-1}(h^{(0)}).$$

In particular \mathcal{H} is a Yetter-Drinfeld module algebra over itself with the action and coaction defined as:

$$g \cdot h = g^{(1)} h S(g^{(2)}), \quad \rho(h) = h^{(1)} \otimes S^{-1}(h^{(0)}).$$

Examples of Yetter-Drinfeld module algebras can be obtained also by considering (co)quasitriangular Hopf algebras. It is shown in [4] that given any \mathcal{H} -algebra (resp. \mathcal{H}^{op} -comodule algebra) A over a quasitriangular (resp. coquasitriangular) Hopf algebra \mathcal{H} , then A can be turned into a Yetter-Drinfeld module algebra. We recall this construction.

By definition, a *quasitriangular Hopf algebra* is a pair (\mathcal{H}, R) , where \mathcal{H} is a Hopf algebra and $R = R^{(1)} \otimes R^{(2)} \in \mathcal{H} \otimes \mathcal{H}$ is an invertible element which satisfies the following relations ($R = r$):

$$\begin{aligned} \Delta(R^{(1)}) \otimes R^{(2)} &= R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}, \quad R^{(1)} \otimes \Delta(R^{(2)}) = R^{(1)} r^{(1)} \otimes R^{(2)} \otimes r^{(2)}, \\ \Delta^{cop}(h) R &= R \Delta(h), \quad \epsilon(R^{(1)}) R^{(2)} = 1, \quad R^{(1)} \epsilon(R^{(2)}) = 1, \\ (S \otimes id) R &= R^{-1}, \quad (id \otimes S) R^{-1} = R, \quad (S \otimes S) R = R. \end{aligned}$$

for all $h \in \mathcal{H}$.

Given any left \mathcal{H} -module algebra A , one can define a right \mathcal{H}^{op} -coaction on A as follows:

$$\rho(a) = R^{(2)} \cdot a \otimes R^{(1)}.$$

It is easily checked that [4], with the above coaction A is a Yetter-Drinfeld \mathcal{H} -algebra.

A *coquasitriangular Hopf algebra* is a pair $(\mathcal{H}, \mathcal{R})$, where \mathcal{H} is a Hopf algebra and $\mathcal{R} \in (\mathcal{H} \otimes \mathcal{H})^*$ is a convolution-invertible map in the sense that there exists a map $\mathcal{R}^{-1} \in (\mathcal{H} \otimes \mathcal{H})^*$ such that

$$\mathcal{R}^{-1}(h^{(0)} \otimes g^{(0)}) \mathcal{R}(h^{(1)} \otimes g^{(1)}) = \mathcal{R}(h^{(0)} \otimes g^{(0)}) \mathcal{R}^{-1}(h^{(1)} \otimes g^{(1)}) = \epsilon(h) \epsilon(g),$$

and the following relations are satisfied:

$$\begin{aligned} \mathcal{R}(hg, r) &= \mathcal{R}(h \otimes r^{(0)}) \mathcal{R}(g \otimes r^{(1)}), \quad \mathcal{R}(h, gr) = \mathcal{R}(h^{(0)} \otimes g) \mathcal{R}(h^{(1)} \otimes r), \\ g^{(0)} h^{(0)} \mathcal{R}(g^{(1)} \otimes h^{(1)}) &= \mathcal{R}(g^{(0)} \otimes h^{(0)}) g^{(1)} h^{(1)}, \quad \mathcal{R}(h \otimes 1) = \mathcal{R}(1 \otimes h) = \epsilon(a), \\ \mathcal{R}(S(h) \otimes g) &= \mathcal{R}^{-1}(h \otimes g), \quad \mathcal{R}^{-1}(h \otimes S(g)) = \mathcal{R}(h \otimes g), \quad \mathcal{R}(S(h) \otimes S(g)) = \mathcal{R}(h \otimes g), \end{aligned}$$

for all $h, g, r \in \mathcal{H}$.

Now for any right \mathcal{H}^{op} -comodule algebra A there is a left \mathcal{H} -module structure on A defined by

$$h \cdot a = a_{<0>} \mathcal{R}(h \otimes a_{<1>}),$$

which turn it into a Yetter-Drinfeld module algebra.

Now let \mathcal{H} be a quasitriangular Hopf algebra. The following lemma shows that the equivariant K -theory of the resulting Yetter-Drinfeld module algebra is independent of the choice of the R -matrix. Recall from [14] that for any \mathcal{H} -module algebra A and an \mathcal{H} -module V there is an

\mathcal{H} -module algebra structure on $A \otimes \text{End}(V)$ where the algebra structure is diagonal and the action is non-diagonal:

$$(h \otimes u)(g \otimes v) = hg \otimes uv, \quad h \cdot (a \otimes u) = h^{(1)} \cdot a \otimes h^{(0)}uS(h^{(2)}), \quad (12)$$

for $a \in A$ and $u \in \text{End}(V)$. We denote this latter structure by $A \bar{\otimes} \text{End}(V)$ and our original \mathcal{H} -algebra structure by $A \otimes \text{End}(V)$ as defined in (10).

Lemma 6.1. *Let A be an \mathcal{H} -algebra over a quasitriangular Hopf algebra and let V be a left \mathcal{H} -module. Then there is an \mathcal{H} -algebra isomorphism between $A \bar{\otimes} \text{End}(V)$ and $A \otimes \text{End}(V)$.*

Proof. We prove that the following maps define \mathcal{H} -isomorphisms between $A \bar{\otimes} \text{End}(V)$ and $A \otimes \text{End}(V)$ inverse to each other:

$$t : A \bar{\otimes} \text{End}(V) \rightarrow A \otimes \text{End}(V), \quad t(a \otimes u) = R^{(2)} \cdot a \otimes R^{(1)}u,$$

$$t' : A \otimes \text{End}(V) \rightarrow A \bar{\otimes} \text{End}(V), \quad t'(a \otimes u) = R^{(2)} \cdot a \otimes S(R^{(1)})u,$$

where R is the R -matrix of \mathcal{H} . Since $t \circ t' = r^{(2)}R^{(2)} \cdot a \otimes S(r^{(1)})R^{(1)}u$ and $t' \circ t = R^{(2)}r^{(2)} \cdot a \otimes S(R^{(1)})r^{(1)}u$ and $(S \otimes id)R = R^{-1}$, therefore $t \circ t' = t' \circ t = id$, where $R = r$. Since

$$\begin{aligned} t(a \otimes u)t(b \otimes w) &= (R^{(2)} \cdot a \otimes R^{(1)}u)(r^{(2)} \cdot b \otimes r^{(1)}w) \\ &= (R^{(2)} \cdot a)((r_1^{(2)}r^{(2)}) \cdot b) \otimes (r_1^{(1)} \cdot (R^{(1)}u)(r^{(1)}w)) \\ &= (R^{(2)} \cdot a)((r_1^{(2)}(\underbrace{r_2^{(2)}r^{(2)}})) \cdot b) \otimes r_1^{(1)}R^{(1)}u \underbrace{S(r_2^{(1)})r^{(1)}}w \\ &= (R^{(2)} \cdot a)(r_1^{(2)} \cdot b) \otimes r_1^{(1)}R^{(1)}uw = R^{(2)} \cdot (ab) \otimes R^{(1)}uw = t((a \otimes u)(b \otimes w)), \end{aligned}$$

where $R = r = r_1 = r_2$, we see that t is an algebra map. Also since

$$t(h \cdot (a \otimes u)) = t(h^{(1)} \cdot a \otimes h^{(0)}uS(h^{(2)})) = (R^{(2)}h^{(1)}) \cdot a \otimes R^{(1)}h^{(0)}uS(h^{(2)}),$$

and $\Delta^{cop}(h)R = R\Delta(h)$, we conclude that

$$t(h \cdot (a \otimes u)) = (h^{(0)}R^{(2)}) \cdot a \otimes h^{(1)}R^{(1)}uS(h^{(2)}) = h^{(0)} \cdot (R^{(2)}a) \otimes h^{(1)} \cdot (R^{(1)}u) = h \cdot t(a \otimes u),$$

which shows that t preserves the \mathcal{H} -actions. \square

A similar result holds in the coquasitriangular case. Motivated by these examples, we were led to the interesting fact that our original \mathcal{H} -algebra structure on $A \otimes \text{End}(V)$ is always independent of the choice of coaction:

Proposition 6.1. *Let A be an \mathcal{H} -Yetter-Drinfeld module algebra and V be a representation of \mathcal{H} . Then there is an \mathcal{H} -algebra isomorphism between $A \bar{\otimes} \text{End}(V)$ with \mathcal{H} -algebra structure defined by (12) and $A \otimes \text{End}(V)$ with \mathcal{H} -algebra structure defined by (10).*

Proof. We prove that the following maps define an \mathcal{H} -isomorphism between $A\bar{\otimes}End(V)$ and $A \otimes End(V)$, inverse to each other:

$$\begin{aligned}\beta : A\bar{\otimes}End(V) &\rightarrow A \otimes End(V), \quad \beta(a \otimes u) = a_{<0>} \otimes a_{<1>}u, \\ \beta' : A \otimes End(V) &\rightarrow A\bar{\otimes}End(V), \quad \beta'(a \otimes u) = a_{<0>} \otimes S(a_{<1>}u).\end{aligned}$$

Since $\beta \circ \beta'(a \otimes u) = a_{<0>} \otimes a_{<1>}S(a_{<2>}u)$ and $\beta' \circ \beta(a \otimes u) = a_{<0>} \otimes S(a_{<1>}a_{<2>}u)$, we obtain $\beta \circ \beta' = \beta' \circ \beta = id$. Now since

$$\begin{aligned}\beta(a \otimes u)\beta(b \otimes w) &= a_{<0>}b_{<0>} \otimes (b_{<1>} \cdot (a_{<1>}u)(b_{<2>}w)) \\ &= a_{<0>}b_{<0>} \otimes b_{<1>}a_{<1>}uS(b_{<2>}b_{<3>}w) = a_{<0>}b_{<0>} \otimes a_{<1>}b_{<1>}uw = \beta((a \otimes u)(b \otimes w)),\end{aligned}$$

β is an algebra map. To check β preserves the \mathcal{H} -actions, we see that

$$\begin{aligned}\beta(h \cdot (a \otimes u)) &= \beta(h^{(1)} \cdot a \otimes h^{(0)}uS(h^{(2)})) = h^{(2)} \cdot a_{<0>} \otimes h^{(3)}a_{<1>}S^{-1}(h^{(1)})h^{(0)}uS(h^{(4)}) \\ &= h^{(0)} \cdot a_{<0>} \otimes h^{(1)}a_{<0>}uS(h^{(2)}) = h \cdot \beta(a \otimes u).\end{aligned}$$

□

The above proposition shows that the two apparently different definitions of $K_0^{\mathcal{H}}$ in this paper and in [14] are in fact the same. Moreover, under the isomorphism β , our generalized trace map (11) transforms to the following map

$$\begin{aligned}(\Psi^n \circ \beta)f(a_0 \otimes u_0, \dots, a_n \otimes u_n)(g) &= \Psi^n f(a_{0<0>} \otimes a_{0<1>}u_0, \dots, a_{n<0>} \otimes a_{n<1>}u_n)(g) \\ &= f(a_{0<0>}, a_{1<0>}, \dots, a_{n<0>})(g^{(1)}) \\ &\quad tr(S(a_{0<1>})a_{0<2>}u_0S(a_{1<1>})a_{1<2>}u_1 \dots S(a_{n<1>})a_{n<2>}u_ng^{(0)}) \\ &= f(a_0, a_1, \dots, a_n)(g^{(1)})tr(u_0u_1 \dots u_ng^{(0)}),\end{aligned} \tag{13}$$

which is the one used in [14]. In fact, the above trace map is exactly the same trace map that we introduced in the first version of this paper for cocommutative Hopf algebras.

Now we show that the quantum monopole line bundle over the Podleś quantum sphere S_q^2 fits very well in our framework and defines a $U_q(su_2)$ -invariant idempotent. Recall that [18] the Podleś equator quantum sphere S_q^2 is the $*$ -algebra generated over \mathbb{C} by the elements a, a^* and b subject to the relations

$$aa^* + q^{-4}b^2 = 1, \quad a^*a + b^2 = 1, \quad ab = q^{-2}ba, \quad a^*b = q^2ba^*.$$

The quantum enveloping algebra $U_q(su_2)$ is a Hopf algebra over \mathbb{C} generated by the elements E, F, K [11] subject to the relations:

$$\begin{aligned}KK^{-1} &= K^{-1}K = 1, \quad KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \quad [F, E] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \\ \Delta(K) &= K \otimes K, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \\ S(K) &= K^{-1}, \quad S(E) = -qE, \quad S(F) = -q^{-1}F, \quad \epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0.\end{aligned}$$

By a direct computation one can show that:

Lemma 6.2. *The following formulas define a $U_q(su_2)$ -module algebra structure on S_q^2 ,*

$$\begin{aligned} K \cdot a &= qa, \quad K \cdot a^* = q^{-1}a^*, \quad K \cdot b = b, \\ E \cdot b &= q^{\frac{5}{2}}a, \quad E \cdot a^* = -q^{\frac{3}{2}}(1 + q^{-2})b, \quad E \cdot a = 0, \\ F \cdot a &= q^{-\frac{7}{2}}(1 + q^2)b, \quad F \cdot b = -q^{-\frac{1}{2}}a^*, \quad F \cdot a^* = 0. \end{aligned}$$

The quantum analogue of the Dirac(or Hopf) monopole line bundle over S^2 is given by the following idempotent in $M_2(S_q^2)$

$$\mathbf{e}_q = \frac{1}{2} \begin{bmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{bmatrix}.$$

It can be directly checked that $\mathbf{e}_q^2 = \mathbf{e}_q$. Now we consider a 2-dimensional representation of $\mathcal{H} = U_q(su_2)$ on $V = \mathbb{C}^2$ defined by [11]

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} q^{-\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{bmatrix}.$$

According to this representation \mathbf{e}_q will be represented as

$$\mathbf{e}_q = \frac{1}{2} \begin{bmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{bmatrix} = \frac{1}{2}(1 \otimes 1 + q^{-2}b \otimes FE - b \otimes EF + qa \otimes F + q^{-1}a^* \otimes E).$$

By a direct computation we show that $\mathbf{e}_q \in M_2(A) = A \bar{\otimes} \text{End}(V)$ is an \mathcal{H} -invariant idempotent, i.e., for all $h \in U_q(su_2)$, $h \cdot \mathbf{e}_q = \epsilon(h)\mathbf{e}_q$. It suffices to check this for the generators E, F and K , i.e., $E \cdot \mathbf{e}_q = 0, F \cdot \mathbf{e}_q = 0, K \cdot \mathbf{e}_q = \mathbf{e}_q$.

Since

$$\begin{aligned} \Delta^2(K) &= K \otimes K \otimes K, \\ \Delta^2(F) &= F \otimes K \otimes K + K^{-1} \otimes F \otimes K + K^{-1} \otimes K^{-1} \otimes F, \\ \Delta^2(E) &= E \otimes K \otimes K + K^{-1} \otimes E \otimes K + K^{-1} \otimes K^{-1} \otimes E, \end{aligned}$$

we can see that

$$\begin{aligned} K \cdot \mathbf{e}_q &= \frac{1}{2} [K \cdot 1 \otimes KK^{-1} + q^{-2}K \cdot b \otimes KFEK^{-1} - K \cdot b \otimes KEFK^{-1} + q \underbrace{K \cdot a}_{qa} \otimes \underbrace{KFK^{-1}}_{q^{-1}F} \\ &\quad + q^{-1} \underbrace{K \cdot a^*}_{q^{-1}a^*} \otimes \underbrace{KEK^{-1}}_{qE}] = \mathbf{e}_q, \end{aligned}$$

and

$$\begin{aligned}
F \cdot \mathbf{e}_q = & \frac{1}{2} [K \cdot 1 \otimes FK^{-1} + \underbrace{q^{-2}K \cdot b \otimes F^2EK^{-1}}_0 - K \cdot b \otimes FEFK^{-1} + \underbrace{qK \cdot a \otimes F^2K^{-1}}_0 + \\
& q^{-1}K \cdot a^* \otimes FEK^{-1} + \underbrace{F \cdot 1 \otimes K^{-2}}_0 + q^{-2}F \cdot b \otimes K^{-1}FEK^{-1} - F \cdot b \otimes K^{-1}EFK^{-1} + \\
& qF \cdot a \otimes K^{-1}FK^{-1} + \underbrace{q^{-1}F \cdot a^* \otimes K^{-1}EK^{-1}}_0 + K^{-1} \cdot 1 \otimes K^{-1}(-q^{-1}F) + \\
& q^{-2}K^{-1} \cdot b \otimes K^{-1}FE(-q^{-1}F) - \underbrace{K^{-1} \cdot b \otimes K^{-1}EF(-q^{-1}F)}_0 + \underbrace{qK^{-1} \cdot a \otimes K^{-1}F(-q^{-1}F)}_0 \\
& + q^{-1}K^{-1} \cdot a^* \otimes K^{-1}E(-q^{-1}F)].
\end{aligned}$$

Since

$$\begin{aligned}
K \cdot 1 \otimes FK^{-1} &= q^{-1}1 \otimes K^{-1}F = q^{-\frac{1}{2}}1 \otimes F \\
K^{-1} \cdot 1 \otimes K^{-1}(-q^{-1}F) &= -q^{-\frac{1}{2}}1 \otimes F \\
-K \cdot b \otimes FEFK^{-1} &= -q^{-1}b \otimes K^{-1}F = -q^{-\frac{1}{2}}b \otimes F \\
q^{-1}K \cdot a^* \otimes FEK^{-1} &= q^{-2}a^* \otimes K^{-1}FE = q^{-\frac{3}{2}}a^* \otimes FE \\
q^{-2}F \cdot b \otimes K^{-1}FEK^{-1} &= -q^{-\frac{5}{2}}a^* \otimes K^{-2}FE = -q^{-\frac{3}{2}}a^* \otimes FE \\
-F \cdot b \otimes K^{-1}EFK^{-1} &= q^{-\frac{1}{2}}a^* \otimes K^{-2}EF = q^{-\frac{3}{2}}a^* \otimes EF, \\
qE \cdot a \otimes K^{-1}FK^{-1} &= q^{-\frac{7}{2}}(1+q^2)b \otimes K^{-2}F = q^{-\frac{5}{2}}(1+q^2)b \otimes F \\
q^{-2}K^{-1} \cdot b \otimes K^{-1}FE(-q^{-1}F) &= -q^{-3}b \otimes K^{-1}F = -q^{-\frac{5}{2}}b \otimes F, \\
q^{-1}K^{-1} \cdot a^* \otimes K^{-1}E(-q^{-1}F) &= -q^{-1}a^* \otimes K^{-1}EF = -q^{-\frac{3}{2}}a^* \otimes EF,
\end{aligned}$$

by adding the above relations, we obtain $F \cdot \mathbf{e}_q = 0$. Also we can see that

$$\begin{aligned}
E \cdot \mathbf{e}_q = & \frac{1}{2} [K \cdot 1 \otimes EK^{-1} + q^{-2}K \cdot b \otimes EFEK^{-1} - \underbrace{K \cdot b \otimes E^2FK^{-1}}_0 + qK \cdot a \otimes EFK^{-1} \\
& + \underbrace{q^{-1}K \cdot a^* \otimes F^2K^{-1}}_0 + \underbrace{E \cdot 1 \otimes K^{-2}}_0 + q^{-2}E \cdot b \otimes K^{-1}FEK^{-1} - E \cdot b \otimes K^{-1}EFK^{-1} + \\
& \underbrace{qE \cdot a \otimes K^{-1}FK^{-1}}_0 + q^{-1}E \cdot a^* \otimes K^{-1}EK^{-1} + K^{-1} \cdot 1 \otimes K^{-1}(-qE) + \\
& \underbrace{q^{-2}K^{-1} \cdot b \otimes K^{-1}FE(-qE)}_0 - K^{-1} \cdot b \otimes K^{-1}EF(-qE) + qK^{-1} \cdot a \otimes K^{-1}F(-qE) + \\
& \underbrace{q^{-1}K^{-1} \cdot a^* \otimes K^{-1}E(-qE)}_0],
\end{aligned}$$

where,

$$\begin{aligned}
K \cdot 1 \otimes EK^{-1} &= q1 \otimes K^{-1}E = q^{\frac{1}{2}}1 \otimes E, \\
q^{-2}K \cdot b \otimes EFEK^{-1} &= q^{-1}b \otimes K^{-1}E = q^{-\frac{3}{2}}b \otimes E, \\
qK \cdot a \otimes EFK^{-1} &= q^2a \otimes K^{-1}EF = q^{\frac{3}{2}}a \otimes EF, \\
q^{-2}E \cdot b \otimes K^{-1}FEK^{-1} &= q^{\frac{1}{2}}a \otimes K^{-2}FE = q^{\frac{3}{2}}a \otimes FE, \\
-E \cdot b \otimes K^{-1}EFK^{-1} &= -q^{\frac{5}{2}}a \otimes K^{-2}EF = -q^{\frac{3}{2}}a \otimes EF, \\
q^{-1}E \cdot a^* \otimes K^{-1}EK^{-1} &= -q^{\frac{3}{2}}(1 + q^{-2})b \otimes K^{-2}E = -q^{\frac{1}{2}}(1 + q^{-2})b \otimes E \\
K^{-1} \cdot 1 \otimes K^{-1}(-qE) - q1 \otimes K^{-1}E &= -q^{\frac{1}{2}}1 \otimes E, \\
-K^{-1} \cdot b \otimes K^{-1}EF(-qE) &= qb \otimes K^{-1}E = q^{\frac{1}{2}}b \otimes E, \\
qK^{-1} \cdot a \otimes K^{-1}F(-qE) &= -qa \otimes K^{-1}FE = -q^{\frac{3}{2}}a \otimes FE,
\end{aligned}$$

and therefore $E \cdot \mathbf{e}_q = 0$.

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